Average Output Power of an Incident Wave Randomly Coupled to a Reflected Wave

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Abstract—Two waves traveling in opposite directions in a lossless waveguide, which are coupled by a random coupling function, are considered. It is assumed that no power is allowed to enter the reflected wave at the output end of the guide. The asymptotic value of the expected output power in the incident wave, when the input power is prescribed, is calculated in the limit of weak coupling and a long guide. The result is compared with that predicted by Marcuse on the basis of coupled power equations. It is found that the two results are quite close, so long as the expected output power is greater than half the input power, but past that point Marcuse's approximation to the asymptotic value becomes increasingly poorer as the waveguide length increases. Some computer simulated results obtained by Marcuse tend to confirm the validity of the asymptotic value of the expected output power.

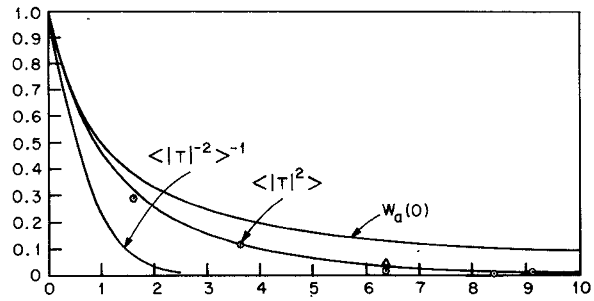

Fig. 1. Average powers $\langle | T |^2 \rangle$ (asymptotic) and $W_a(0)$ (approximate), and $\langle | T |^{-2} \rangle^{-1}$ (asymptotic), versus normalized waveguide length s. Computer simulated results are indicated by circles and by a triangle.

I. Introduction

In a recent paper, Marcuse [1] has considered the problem of two waves traveling in opposite directions that are coupled by a random coupling function. He derived equations for the average powers in the two waves from the coupled wave equations, making use of perturbation approximations, and some intuitive assumptions, it being assumed that the coupling is sufficiently weak. He found that the form of the coupled power equations depends on the boundary conditions imposed. One set of equations describes the situation in which the output amplitude of the incident wave is prescribed, while no power is allowed to enter the reflected wave at the output end. Another set of equations describes the situation in which the input amplitude of the incident wave is prescribed, while again no power is allowed to enter the reflected wave at the output end.

The former case, in which the amplitudes of both waves are prescribed at the output end, is simpler. The coupled power equations derived by Marcuse agree with the kinetic power equations derived by Papanicolaou [2], in the case of weak coupling and a long guide, using a method developed by Papanicolaou and Keller [3]. In the latter case, in which the amplitudes of the two waves are prescribed at opposite ends of the guide, the kinetic power equations derived by Papanicolaou are no longer applicable.

In this paper we investigate the validity of the coupled power equations derived by Marcuse in the latter case, in so far as they predict the expected value of the output power when the input power is unity. Specifically, we apply a limit theorem due to Khas'minskii [4] in order to calculate the asymptotic value of the expected output power, in the limit of weak coupling and a long guide. The results are discussed in detail in Section IV, and are depicted graphically in Fig. 1. The quantity $W_a(0)$ is the expected value of the output power predicted by Marcuse [1], while $\langle | T |^2 \rangle$ is the asymptotic value. The parameter s is an appropriate normalization of the waveguide length. It is seen that $W_a(0)$ is quite close to $\langle | T |^2 \rangle$, so long as it exceeds half the input power, but past that point the approximation to the asymptotic value becomes increasingly poor, as the waveguide length increases. Also depicted in Fig. 1 is the asymptotic value of $\langle | T |^{-2} \rangle^{-1}$, which is the reciprocal of the expected value of the input power, when the output power is unity.

Comparison is also made with some computer simulated results obtained by Marcuse [1], and these tend to confirm the validity of the asymptotic value of the expected output power, derived on the basis of the limit theorem.

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II. Coupled Wave Equations

Marcuse [1] considers coupled wave equations of the form

$$\frac{da}{dt} = i\beta_1 a - c(t) b \tag{1}$$

$$\frac{db}{dt} = -i\beta_2 b - c^*(t) a \tag{2}$$

where the propagation constants β_1 and β_2 are assumed to be positive, so that losses in the waveguide are neglected. For convenience, we have replaced Marcuse's variable z, measuring distance along the guide, by $(L - t)$, where L is the length of the guide. The wave of amplitude

a travels in the negative t direction with propagation constant β_1, while the wave with amplitude b travels in the positive t direction with propagation constant β_2.

We write the complex coupling coefficient in the form

$$c(t) = \epsilon\eta(t) = \epsilon[\eta_1(t) + i\eta_2(t)] \tag{3}$$

where $\epsilon > 0$ is a small parameter, so that the coupling between the waves is weak. The asterisk in (2) denotes complex conjugate. We assume that $c(t)$ is a zero-mean wide sense stationary stochastic process, so that

$$\langle \eta_j(t) \rangle = 0, \quad j = 1,2 \tag{4}$$

where $\langle\ \rangle$ denotes ensemble average, and the correlation functions are

$$\langle \eta_j(t)\eta_k(s) \rangle = \Gamma_{jk}(t - s), \qquad j,k = 1,2. \tag{5}$$

Note that

$$\Gamma_{jk}(-\zeta) = \Gamma_{kj}(\zeta), \qquad j,k = 1,2 \tag{6}$$

and that

$$r(\zeta) \equiv \langle \eta(s+\zeta)\eta^*(s) \rangle = \langle \eta(s)\eta^*(s-\zeta) \rangle = r^*(-\zeta). \tag{7}$$

Marcuse [1] assumes that $r(-\zeta) = r(\zeta)$, which implies that $r(\zeta)$ is real and that $\Gamma_{12}(\zeta) = \Gamma_{21}(\zeta)$, but we do not impose this restriction here.

It follows from (1) and (2) that

$$\frac{d}{dt}[\,|a(t)|^2 - |b(t)|^2] = 0 \tag{8}$$

which expresses the conservation of power. It is assumed that the input power is in the wave of amplitude a, and that no power enters the reflected wave at $t = 0$, the output end, so that

$$b(0) = 0. \tag{9}$$

Marcuse [1] derived coupled power equations for the average powers in the two waves

$$W_a(t) = \langle |a(t)|^2 \rangle, \qquad W_b(t) = \langle |b(t)|^2 \rangle \tag{10}$$

in two different cases. In one case he assumed that the output amplitude of the incident wave is prescribed. In view of (9), there is then an initial value problem for the system (1) and (2).

Kinetic power equations governing the power transfer between randomly coupled modes, subject to nonstochastic initial conditions, have been obtained by Papanicolaou [2] in the case of weak coupling and long guides, using a method developed earlier by him and Keller [3]. For the particular case of systems (1) and (2), these equations take the form

$$\frac{dW_a}{dt} = \epsilon^2\delta(W_a + W_b) = \frac{dW_b}{dt} \tag{11}$$

where

$$\delta = \int_{-\infty}^{\infty} e^{-i\gamma\zeta} r(\zeta)\, d\zeta \tag{12}$$

and

$$\gamma = \beta_1 + \beta_2. \tag{13}$$

In (12), $r(\zeta)$ is given by (7), and it is known [5] that $\delta \geq 0$. Equation (11) is asymptotically valid for $t = \tau/\epsilon^2$, where $0 \leq \tau \leq \tau_0$, and $\tau_0 > 0$ is fixed, in the limit $\epsilon \to 0$. They are consistent with the coupled power equations obtained by Marcuse [1] in the case $r(-\zeta) = r(\zeta)$. The solution of (11), subject to the initial conditions $W_a(0) = 1$, $W_b(0) = 0$ is

$$W_a(t) = \tfrac{1}{2}(e^{2\epsilon^2\delta t}+1)$$
$$W_b(t) = \tfrac{1}{2}(e^{2\epsilon^2\delta t}-1). \tag{14}$$

In the other case considered by Marcuse [1] the input amplitude of the incident wave is prescribed, say $a(L) = 1$, so that, in view of (9), there is then a two-point boundary value problem for systems (1) and (2), rather than an initial value problem. The kinetic power equations of Papanicolaou [2] are not applicable in this case. However, Marcuse [1] does derive some coupled power equations with the help of some intuitive assumptions, and they take the form

$$\frac{dW_a}{dt} = \epsilon^2\delta(W_a - W_b) = \frac{dW_b}{dt} \tag{15}$$

where δ is given by (12), with $r(\zeta) = r(-\zeta)$ given by (7). The solution of (15) subject to the boundary conditions $W_a(L) = 1$, $W_b(0) = 0$ is

$$W_a(t) = \frac{(1+\epsilon^2\delta t)}{(1+\epsilon^2\delta L)} \qquad W_b(t) = \frac{\epsilon^2\delta t}{(1+\epsilon^2\delta L)}. \tag{16}$$

The expected value of the output power is then

$$W_a(0) = \frac{1}{(1+\epsilon^2\delta L)}. \tag{17}$$

The main purpose of this paper is to investigate the validity of (17) in the case $0 \leq L \leq l_0/\epsilon^2$, where ϵ is small. We do this with the help of a limit theorem due to Khas'minskii [4], which is discussed in the next section. However, we first reformulate the problem in a manner analogous to that adopted by Papanicolaou [6] in his investigation of the mean power transmitted by an electromagnetic wave normally incident on a randomly stratified dielectric slab. Thus we let

$$\rho(t) = e^{i\gamma t}\frac{b(t)}{a(t)} \tag{18}$$

where γ is given by (13). Then, from (1)–(3), it follows that

$$\frac{d\rho}{dt} = \epsilon[e^{-i\gamma t}\eta(t)\rho^2 - e^{i\gamma t}\eta^*(t)]. \tag{19}$$

From (9) the initial condition is

$$\rho(0) = 0. \tag{20}$$

From (8), (9), and (18) it follows that

$$(1 - |\rho(t)|^2) = \frac{|a(0)|^2}{|a(t)|^2}. \tag{21}$$

Hence, if $a(L) = 1$, the output power $|a(0)|^2$ is

$$|T|^2 = (1 - |\rho(L)|^2). \tag{22}$$

Note also from (21) that in the other case considered by Marcuse, in which $|a(0)|^2 = 1$, $|\rho(t)|$ determines $|a(t)|$. Thus, for our purposes, it suffices to consider (19), subject to the initial condition (20). Although (19) differs from the Riccati equation that arises in the slab problem [6], we are able to make use of the main result for that problem, as will be seen in the next section.

III. Application of a Limit Theorem

For the application of the limit theorem of Khas'minskii [4], it is necessary to separate (19) into real and imaginary parts. Thus, letting

$$\rho(t) = \sigma(t)\, e^{i\psi(t)} \tag{23}$$

it is found, using (3), that

$$\frac{d\sigma}{dt} = \epsilon(\sigma^2 - 1)[\eta_1(t)\cos(\psi - \gamma t) - \eta_2(t)\sin(\psi - \gamma t)] \tag{24}$$

$$\frac{d\psi}{dt} = \epsilon\frac{(\sigma^2 + 1)}{\sigma}[\eta_1(t)\sin(\psi - \gamma t) + \eta_2(t)\cos(\psi - \gamma t)]. \tag{25}$$

The limit theorem applies to vector differential equations of the form

$$\frac{d\psi_j}{dt} = F_j(\boldsymbol{\psi},t) \tag{26}$$

where $F_j(\boldsymbol{\psi},t)$ is a random function of t, with

$$\langle F_j(\boldsymbol{\psi},t)\rangle = 0. \tag{27}$$

It is supposed that the initial value $\boldsymbol{\psi}(0)$ is nonstochastic. In view of (4), systems (24) and (25) are of the appropriate form, with

$$\boldsymbol{\psi} = (\psi_1,\psi_2) = (\sigma,\psi). \tag{28}$$

Now define

$$K_j(\boldsymbol{\psi},s,t) = \left\langle \frac{\partial F_j}{\partial \psi_k}(\boldsymbol{\psi},s)\, F_k(\boldsymbol{\psi},t)\right\rangle \tag{29}$$

where it is understood that repeated indices are summed, and

$$a_{jk}(\boldsymbol{\psi},s,t) = \langle F_j(\boldsymbol{\psi},s)\, F_k(\boldsymbol{\psi},t)\rangle. \tag{30}$$

For systems (24) and (25), it follows from (5) that

$$K_j\left(\boldsymbol{\psi},s + \frac{2\pi}{\gamma}, t + \frac{2\pi}{\gamma}\right) = K_j(\boldsymbol{\psi},s,t) \tag{31}$$

and

$$a_{jk}\left(\boldsymbol{\psi},s + \frac{2\pi}{\gamma}, t + \frac{2\pi}{\gamma}\right) = a_{jk}(\boldsymbol{\psi},s,t). \tag{32}$$

In view of (31) and (32), Khas'minskii's definitions of $\bar{K}_j$ and $\bar{a}_{jk}$ become [4]

$$\bar{K}_j(\boldsymbol{\psi}) = \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma}\int_0^{\infty} K_j(\boldsymbol{\psi},s,s - \zeta)\, d\zeta\, ds \tag{33}$$

$$\bar{a}_{jk}(\boldsymbol{\psi}) = \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma}\int_{-\infty}^{\infty} a_{jk}(\boldsymbol{\psi},s,s - \zeta)\, d\zeta\, ds. \tag{34}$$

We assume that $\eta(t)$, as given by (3), is a bounded stochastic process satisfying a certain strong mixing condition. The reader is referred to Khas'minskii's paper for a precise statement of this condition. Let $\boldsymbol{\psi}(t)$ be the solution of (26) satisfying $\boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$. Then the limit theorem states that on the interval $0 \le \tau < \tau_0$, where τ_0 is an arbitrary positive number, the process $\boldsymbol{\psi}(\tau/\epsilon^2)$ converges weakly as $\epsilon \to 0$ to a Markov process $\boldsymbol{\xi}(\tau)$, which is continuous with probability 1 and whose infinitesimal generator is given by

$$A \equiv \frac{1}{2}\bar{a}_{jk}(\boldsymbol{\xi})\frac{\partial^2}{\partial\xi_j\partial\xi_k} + \bar{K}_j(\boldsymbol{\xi})\frac{\partial}{\partial\xi_j}. \tag{35}$$

If $g(\tau, \boldsymbol{\xi})$ satisfies the backward equation

$$\frac{\partial g}{\partial \tau} = A[g] \tag{36}$$

with initial condition

$$g(0,\boldsymbol{\xi}) = G(\boldsymbol{\xi}) \tag{37}$$

then [7]

$$g(\tau,\boldsymbol{\xi}(0)) = \langle G(\boldsymbol{\xi}(\tau))\rangle. \tag{38}$$

If $G(\boldsymbol{\xi})$ is bounded and continuous, the weak convergence of $\boldsymbol{\psi}(\tau/\epsilon^2)$ to $\boldsymbol{\xi}(\tau)$ implies [5, p. 243] that

$$\langle G(\boldsymbol{\xi}(\epsilon^2 t))\rangle - \langle G(\boldsymbol{\psi}(t))\rangle \to 0 \tag{39}$$

as $\epsilon \to 0$, for $0 \le \epsilon^2 t < \tau_0$.

It is a straightforward matter to calculate the quantities $\bar{K}_j(\boldsymbol{\psi})$ and $\bar{a}_{jk}(\boldsymbol{\psi})$ corresponding to systems (24) and (25), so we omit the details. The integrals over s in (33) and (34) are readily carried out, and it is found that

$$\bar{K}_1 = \mu(\sigma^2 - 1)^2/2\sigma \qquad \bar{K}_2 = 2\nu \tag{40}$$

$$\bar{a}_{11} = \mu(\sigma^2 - 1)^2 \qquad \bar{a}_{22} = \mu(\sigma^2 + 1)^2/\sigma^2 \tag{41}$$

and

$$\bar{a}_{12} = 0 \quad \bar{a}_{21} = 0 \tag{42}$$

where, in terms of the correlation function (7),

$$\mu - i\nu = \int_0^{\infty} e^{-i\gamma\zeta} r(\zeta)\, d\zeta. \tag{43}$$

In deriving these results we have made use of (3), (5), and (6). Note that, from (7) and (12),

$$2\mu = \int_0^\infty [e^{-i\gamma\zeta}r(\zeta) + e^{i\gamma\zeta}r^*(\zeta)]\,d\zeta = \delta. \quad (44)$$

We remark that, strictly speaking, Khas'minskii's theorem is not applicable to systems (24) and (25), because of the singularity in (25) at $\sigma = 0$. However, one may let $\rho = u + iv$ in (19), and apply the theorem to the equations satisfied by u and v. It may be verified that the corresponding infinitesimal generator is that obtained by making the corresponding transformation of variables in (35). In the (σ, ψ) variables, (36) becomes, from (40)–(42),

$$\frac{\partial g}{\partial \tau} = \frac{\mu}{2}(1-\sigma^2)^2\left(\frac{1}{\sigma}\frac{\partial g}{\partial \sigma} + \frac{\partial^2 g}{\partial \sigma^2}\right) + 2\nu\frac{\partial g}{\partial \psi} + \frac{\mu(1+\sigma^2)^2}{2\sigma^2}\frac{\partial^2 g}{\partial \psi^2}. \quad (45)$$

If we now let

$$\sigma^2 = \frac{x-1}{x+1} \quad (46)$$

then (45) becomes

$$\frac{1}{2}\frac{\partial g}{\partial \tau} = \mu\frac{\partial}{\partial x}\left[(x^2-1)\frac{\partial g}{\partial x}\right] + \nu\frac{\partial g}{\partial \psi} + \frac{\mu x^2}{(x^2-1)}\frac{\partial^2 g}{\partial \psi^2}. \quad (47)$$

We are interested only in

$$1 - |\rho|^2 = 1 - \sigma^2 = \frac{2}{x+1} \quad (48)$$

from (23) and (46). Then G is a function of x alone and, from (37) and (47), g is independent of ψ. From (20), (38), (39), and (48),

$$\langle G(\psi(t))\rangle \sim g(\epsilon^2 t, 1). \quad (49)$$

Since $|\rho| \leq 1$ from (21), it follows that $x \geq 1$.

The solution of (47), with initial value given by (48), has been obtained previously [3], [6], [8], [9] in the calculation of the mean power transmitted through a randomly stratified dielectric slab. We comment that in the slab problem the equation corresponding to (47) contains an additional term [9], which is a constant multiple of $\partial^2 g/\partial\psi^2$. From (22), (48), and (49), with

$$s = 2\epsilon^2\mu L = \epsilon^2\delta L \quad (50)$$

from (44), it follows that [8], [9]

$$\langle |T|^2\rangle \sim \frac{4}{\sqrt{\pi}}e^{-s/4}\int_0^\infty \frac{v^2\,e^{-v^2}\,dv}{\cosh(\sqrt{s}\,v)}. \quad (51)$$

There is an unfortunate misprint in Papanicolaou's result (the exponential factor in front of the integral in [6, eq. (3.14)] should be as in [6, eq. (3.10)]).

We will discuss the result in (51) in the next section, but now turn to the case considered by Marcuse [1] in which $|a(0)|^2 = 1$. Then, from (21),

$$|a(t)|^2 = \frac{1}{(1 - |\rho(t)|^2)}. \quad (52)$$

The corresponding initial value of g is now

$$g(0, x) = G(x) = \tfrac{1}{2}(1 + x). \quad (53)$$

Since $G(x)$ is now unbounded, (39) does not necessarily follow. However, the solution of (47), subject to the initial condition (53), is readily found to be

$$g(\tau, x) = \tfrac{1}{2}(1 + xe^{4\mu\tau}). \quad (54)$$

Thus from (48), (49), and (52)–(54),

$$\langle |a(t)|^2\rangle \sim \tfrac{1}{2}(1 + e^{4\epsilon^2\mu t}). \quad (55)$$

This agrees with (14), in view of (10) and (44).

IV. Comparison of Results

The quantity $\langle |T|^2\rangle$ is the expected value of the output power when the input power is unity. For small ϵ the lowest order approximation to $\langle |T|^2\rangle$ is given by (51), where $s = \epsilon^2\delta L = 0(1)$, and δ is given by (12). The expression in (51) has been calculated numerically [3], [6], [8], [9], and it is depicted graphically in Fig. 1. The corresponding quantity according to Marcuse's coupled power equations is given, from (17), by

$$W_a(0) = \frac{1}{(1+s)} \quad (56)$$

and is also depicted in Fig. 1. It is seen that the two are quite close, so long as the expected output power exceeds half the input power, but that past that point ($s \gtrsim 1$) the curves diverge, considerably so for large s.

Marcuse [1] carried out some computer simulated experiments, in which he considered N sections of guide of equal length D, so that $L = ND$. Within each section he took the coupling coefficient c to be $\pm\kappa$, with κ constant, but the signs chosen at random. He obtained the average output power $P_a(L)$, corresponding to unit input power, by averaging over 10 such random waveguides. For simplicity he assumed that $\beta_1 = \beta = \beta_2$ and gave numerical results in the case $\beta D = \pi/4$. We may take $\epsilon = \kappa/\beta$. Then, from [1, eq. (28)], since $\cos 2\beta D = 0$, we have

$$s = \frac{\epsilon^2 L}{2D} = \frac{1}{2}N\epsilon^2 = \frac{N}{8\pi^2}\left(\frac{2\pi\kappa}{\beta}\right)^2. \quad (57)$$

Marcuse [1] gives the experimental values of $P_a(L)$ in the case $N = 500$ for three values of κ/β. For $2\pi\kappa/\beta = 1$, which corresponds to $s = 6.33$, he finds that $P_a(L) = 0.0496$. This point is indicated by a triangle in Fig. 1, and it is seen that it lies quite close to the $\langle |T|^2\rangle$ curve. The other two values of κ/β correspond to $s = 0.063$ and $s = 0.57$, and the corresponding values of $P_a(L)$ lie close to both the $\langle |T|^2\rangle$ and $W_a(0)$ curves. Marcuse [1] also gives experimental values of $P_a(L)$ in the case $N = 100$

for four values of κ/β. Three of these values correspond to $s < 1.3$, and the corresponding values of $P_a(L)$ lie close to both the $\langle | T |^2 \rangle$ and $W_a(0)$ curves. The fourth value of $2\pi\kappa/\beta$ is 3 (for which ϵ is not very small), which corresponds to $s = 11.4$, and lies outside the range of s in Fig. 1.

Since the above experimental values give only one comparison point in the region in which $W_a(0)$ and $\langle | T |^2 \rangle$ are not close, Marcuse [10], at the request of the author, carried out some more computer simulated experiments in the case $N = 500$. The additional points are indicated by circles in Fig. 1. The values for $s = 8.37$ and $s = 9.12$ were obtained by averaging over 40 waveguides, rather than 10, because of the large scatter for these values of s. The experimental values tend to confirm the validity of the asymptotic value $\langle | T |^2 \rangle$ of the average output power, given by (51), subject to (12) and (50). We emphasize that the asymptotic result holds for quite general weak zero-mean wide sense stationary coupling.

Also depicted in Fig. 1 is the quantity $\langle | T |^{-2} \rangle^{-1}$. From (21) and (22), note that $| T |^{-2}$ is the value of the input power $| a(L) |^2$, when the output power $| a(0) |^2$ is unity. From (50), (52), and (55), it follows that

$$\langle | T |^{-2} \rangle^{-1} = \frac{2}{(1 + e^{2s})}. \tag{58}$$

Although $\langle | T |^{-2} \rangle^{-1}$ has the same initial slope as $\langle | T |^2 \rangle$, it is seen that it decreases very much more rapidly with increasing s.

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Short Papers

Ridged Circular Waveguide

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Abstract—**Characteristics of wave propagation inside a ridged circular waveguide are studied. The waveguide is a hollow, conducting circular cylinder with a pair of semicircular conducting ridges diametrically attached to its inside wall. Results of a perturbation analysis suggest that in this device a lower attenuation and a wider bandwidth than those of a conventional circular waveguide can be achieved. Certain numerical results are graphically presented.**

INTRODUCTION

It has been found experimentally that, in a seam-weld circular waveguide, the polarization of the dominant TE_{11} wave or the orientation of the line joining the two E_ρ maxima wanders, and the wave has a great tendency to orient its E_ρ maximum along the seam ridges [1]. Further experimentation indicated that this tendency was pronounced for a thick seam or with a seam intended inward to form a semicircular ridge. In this study, an analysis is made of the characteristic of wave propagation in a hollow, conducting circular cylinder with a pair of semicircular conducting longitudinal ridges diametrically attached to its inside wall.

To solve this problem as an exact boundary-value problem, the process is long and quite complicated. Instead, a formulation based on perturbation theory is used. The result is expected to be quite good for small ridges with smooth cross sections.

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CUTOFF FREQUENCY

Let us consider a ridged circular waveguide with its longitudinal axis in the z direction of a cylindrical coordinate system. The symmetric ridge-pair assumed to have a semicircular cross section is shown in Fig. 1, where θ is defined as the angle between the longitudinal plane bisecting the ridge-pair (ridge-pair plane) and the longitudinal plane containing the two E_ρ maxima (polarization).

The ridged circular waveguide may be considered as a smooth waveguide with its boundary wall perturbed by a symmetric ridge-pair along the longitudinal direction. It is well known that the time average of stored magnetic and electric energies are equal in a waveguide at cutoff frequency. A small deformation in the waveguide wall will cause an unbalance in these energies. Therefore, the cutoff frequency will have to shift by an amouut necessary to reequalize